1.3 Exponential Functions

You will be able to model exponential growth and decay with functions of the form $y = k \cdot a^x$ and recognize exponential growth and decay in algebraic, numerical, and graphical representations.

- Exponential functions
- Rules of exponents
- Applications of exponential functions (growth and decay)
- Compound interest
- The number e

Exponential Growth

Table 1.1 shows the growth of \$100 invested in 1996 at an interest rate of 5.5%, compounded annually.

TABLE 1.1 Savings Account Growth

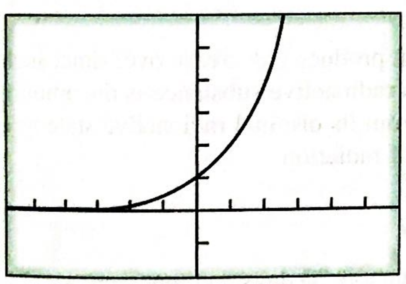
Year	Amount (dollars)	Increase (dollars)
1996	100	
1997	$100(1.055) = 105.50$	5.50
1998	$100(1.055)^2 = 111.30$	5.80
1999	$100(1.055)^3 = 117.42$	6.12
2000	$100(1.055)^4 = 123.88$	6.46

After the first year, the value of the account is always 1.055 times its value in the previous year. After n years, the value is $y = 100 \cdot (1.055)^n$.

Compound interest provides an example of *exponential growth* and is modeled by a function of the form $y = P \cdot a^x$, where P is the initial investment (called the *principal*) and a is equal to 1 plus the interest rate expressed as a decimal.

The equation $y = P \cdot a^x$, $a > 0$, $a \neq 1$, identifies a family of functions called *exponential functions.* Notice that the ratio of consecutive amounts in Table 1.1 is always the same: $111.30/105.30 = 117.42/111.30 = 123.88/117.42 \approx 1.055$. This fact is an important feature of exponential curves that has widespread application, as we will see.

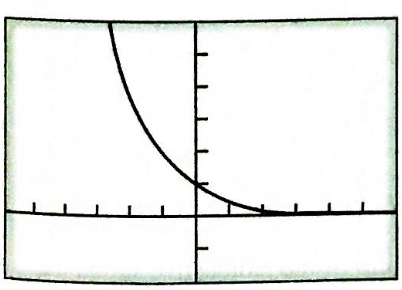
$y = 2^x$



$[-6, 6]$ by $[-2, 6]$

(a)

$y = 2^{-x}$



$[-6, 6]$ by $[-2, 6]$

(b)

Figure 1.20 A graph of (a) $y = 2^x$ and (b) $y = 2^{-x}$.

EXPLORATION 1 Exponential Functions

1. Graph the function $y = a^x$ for $a = 2, 3, 5$, in a $[-5, 5]$ by $[-2, 5]$ viewing window.
2. For what values of x is it true that $2^x < 3^x < 5^x$?
3. For what values of x is it true that $2^x > 3^x > 5^x$?
4. For what values of x is it true that $2^x = 3^x = 5^x$?
5. Graph the function $y = (1/a)^x = a^{-x}$ for $a = 2, 3, 5$.
6. Repeat parts 2–4 for the functions in part 5.

DEFINITION Exponential Function

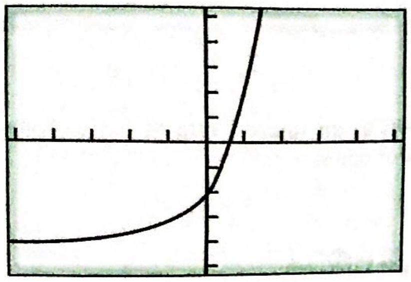
Let a be a positive real number other than 1. The function

$$f(x) = a^x$$

is the **exponential function with base a.**

The domain of $f(x) = a^x$ is $(-\infty, \infty)$ and the range is $(0, \infty)$. If $a > 1$, the graph of f looks like the graph of $y = 2^x$ in Figure 1.20a. If $0 < a < 1$, the graph of f looks like the graph of $y = 2^{-x}$ in Figure 1.20b.

$y = 2(3^x) - 4$



[−5, 5] by [−5, 5]

Figure 1.21 The graph of $y = 2(3^x) - 4$. (Example 1)

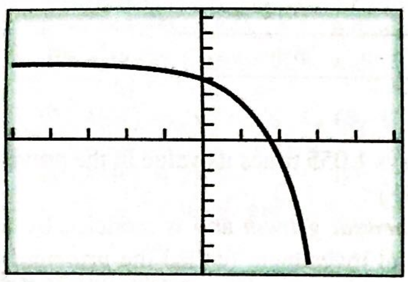
EXAMPLE 1 Graphing an Exponential Function

Graph the function $y = 2(3^x) - 4$. State its domain and range.

SOLUTION

Figure 1.21 shows the graph of the function y. It appears that the domain is $(-\infty, \infty)$. The range is $(-4, \infty)$ because $2(3^x) > 0$ for all x. ***Now Try Exercise 1.***

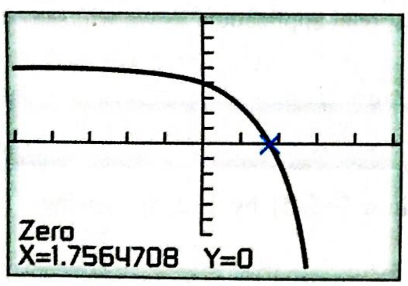
$y = 5 - 2.5^x$



[−5, 5] by [−8, 8]

(a)

$y = 5 - 2.5^x$



[−5, 5] by [−8, 8]

(b)

Figure 1.22 (a) A graph of $f(x) = 5 - 2.5^x$. (b) Showing the use of the ZERO feature to approximate the zero of f. (Example 2)

EXAMPLE 2 Finding Zeros

Find the zeros of $f(x) = 5 - 2.5^x$ graphically.

SOLUTION

Figure 1.22a suggests that f has a zero between $x = 1$ and $x = 2$, closer to 2. We can use our grapher to find that the zero is approximately 1.756 (Figure 1.22b). ***Now Try Exercise 9.***

Exponential functions obey the rules for exponents.

Rules for Exponents

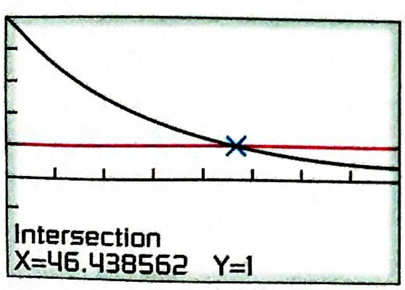
If $a > 0$ and $b > 0$, the following hold for all real numbers x and y.

1. $a^x \cdot a^y = a^{x+y}$
2. $\dfrac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = (a^y)^x = a^{xy}$
4. $a^x \cdot b^x = (ab)^x$
5. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

Exponential Decay

Exponential functions can also model phenomena that produce a decrease over time, such as happens with radioactive decay. The **half-life** of a radioactive substance is the amount of time it takes for half of the substance to change from its original radioactive state to a nonradioactive state by emitting energy in the form of radiation.

$y = 5\left(\frac{1}{2}\right)^{t/20}$, $y = 1$



[0, 80] by [−3, 5]

Figure 1.23 (Example 3)

EXAMPLE 3 Modeling Radioactive Decay

Suppose the half-life of a certain radioactive substance is 20 days and that there are 5 grams present initially. When will there be only 1 gram of the substance remaining?

SOLUTION

The number of grams remaining after 20 days is

$$5\left(\frac{1}{2}\right) = \frac{5}{2}.$$

The number of grams remaining after 40 days is

$$5\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 5\left(\frac{1}{2}\right)^2 = \frac{5}{4}.$$

The function $y = 5(1/2)^{t/20}$ models the mass in grams of the radioactive substance after t days. Figure 1.23 shows that the graphs of $y_1 = 5(1/2)^{t/20}$ and $y_2 = 1$ (for 1 gram) intersect when t is approximately 46.44.

There will be 1 gram of the radioactive substance left after approximately 46.44 days, or about 46 days 10.5 hours. ***Now Try Exercise 23.***

Compound interest investments, population growth, and radioactive decay are all examples of *exponential growth and decay.*

DEFINITIONS Exponential Growth, Exponential Decay

The function $y = k \cdot a^x, k > 0$ is a model for **exponential growth** if $a > 1$, and a model for **exponential decay** if $0 < a < 1$.

Compound Interest

One common application of exponential growth in the financial world is the compounding effect of accrued interest in a savings account. We saw at the beginning of this section that an account paying 5.5% annual interest would multiply by a factor of 1.055 every year, so an account starting with principal P would be worth $P(1.055)^t$ after t years.

Some accounts pay interest multiple times per year, which increases the compounding effect. An account earning 6% annual interest compounded monthly would pay one-twelfth of the interest each month, effectively a monthly interest rate of 0.5%. Thus, after t years, the account would be worth $P(1.005)^{12t}$. The general formula is given below.

Compound Interest Formula

If an account begins with principal P and earns an annual interest rate r compounded n times per year, then the value of the account in t years is

$$P\left(1 + \frac{r}{n}\right)^{nt}.$$

EXAMPLE 4 Compound Interest

Bernie deposits \$2500 in an account earning 5.4% annual interest. Find the value of the account in 10 years if the interest is compounded

(a) annually; **(b)** quarterly; **(c)** monthly.

SOLUTION

(a) $2500(1 + 0.054)^{10} \approx 4230.06$, so the account is worth \$4230.06.

(b) $2500\left(1 + \frac{0.054}{4}\right)^{4(10)} \approx 4274.55$, so the account is worth \$4274.55.

(c) $2500\left(1 + \frac{0.054}{12}\right)^{12(10)} \approx 4284.82$, so the account is worth \$4284.82.

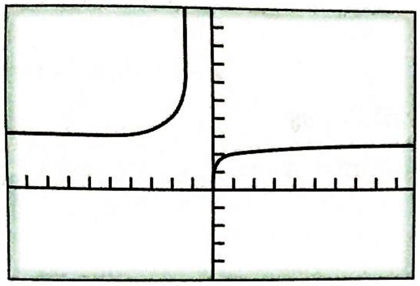
Now Try Exercises 25 and 26.

The Number *e*

You may have noticed in Example 4 that the value of the account becomes greater as the number of interest payments per year increases, but the value does not appear to be increasing without bound. In fact, a bank can even offer to compound the interest *continuously* without paying that much more per year. This is because the number that serves as the base of the exponential function in the compound interest formula has a finite limit as n approaches infinity:

$$\left(1 + \frac{r}{n}\right)^n \to e^r \quad \text{as} \quad n \to \infty$$

$y = (1 + 1/x)^x$



[−10, 10] by [−5, 10]

X	Y_1
1000	2.7169
2000	2.7176
3000	2.7178
4000	2.7179
5000	2.718
6000	2.7181
7000	2.7181

$Y_1 = (1+1/X)$^X

Figure 1.24 A graph and table of values for $f(x) = (1 + 1/x)^x$ both suggest that as $x \to \infty$, $f(x) \to e \approx 2.718$.

You can get an estimate of the number e by looking at values of $\left(1 + \frac{1}{x}\right)^x$ for increasing values of x, as shown graphically and numerically in Figure 1.24. This number e, which is approximately 2.71828, is one of the most important numbers in mathematics. Like π, it is an irrational number that shows up in many different contexts, some of them quite surprising. You will encounter several of them in this course.

One place that e appears is in the formula for continuously compounded interest.

Continuously Compounded Interest Formula

If an account begins with principal P and earns an annual interest rate r compounded continuously, then the value of the account in t years is

$$P(e^r)^t = Pe^{rt}.$$

EXAMPLE 5 Continuously Compounded Interest

Bernice deposits \$2500 in an account earning 5.4% annual interest. Find the value of the account in 10 years if the interest is compounded

(a) daily; **(b)** continuously.

SOLUTION

(a) $2500\left(1 + \frac{0.054}{365}\right)^{365(10)} \approx 4289.85$, so the account is worth \$4289.85.

(b) $2500e^{0.054(10)} \approx 4290.02$, so the account is worth \$4290.02.

Notice that the difference between "daily" and "continuously" for Bernice is 17 cents!

Now Try Exercise 27.

Quick Review 1.3 *(For help, go to Section 1.3.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1–3, evaluate the expression. Round your answers to 3 decimal places.

1. $5^{2/3}$ **2.** $3^{\sqrt{2}}$

3. $3^{-1.5}$

In Exercises 4–6, solve the equation. Round your answers to 4 decimal places.

4. $x^3 = 17$ **5.** $x^5 = 24$

6. $x^{10} = 1.4567$

In Exercises 7 and 8, find the value of investing P dollars for n years with the interest rate r compounded annually.

7. $P = \$500$, $r = 4.75\%$, $n = 5$ years

8. $P = \$1000$, $r = 6.3\%$, $n = 3$ years

In Exercises 9 and 10, simplify the exponential expression.

9. $\dfrac{(x^{-3}y^2)^2}{(x^4y^3)^3}$ **10.** $\left(\dfrac{a^3b^{-2}}{c^4}\right)^2\left(\dfrac{a^4c^{-2}}{b^3}\right)^{-1}$

Section 1.3 Exercises

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1–4, graph the function. State its domain and range.

1. $y = -2^x + 3$ **2.** $y = e^x + 3$

3. $y = 3 \cdot e^{-x} - 2$ **4.** $y = -2^{-x} - 1$

In Exercises 5–8, rewrite the exponential expression to have the indicated base.

5. 9^{2x}, base 3 **6.** 16^{3x}, base 2

7. $(1/8)^{2x}$, base 2 **8.** $(1/27)^x$, base 3

In Exercises 9–12, use a graph to find the zeros of the function.

9. $f(x) = 2^x - 5$

10. $f(x) = e^x - 4$

11. $f(x) = 3^x - 0.5$

12. $f(x) = 3 - 2^x$

In Exercises 13–18, match the function with its graph. Try to do it without using your grapher.

13. $y = 2^x$

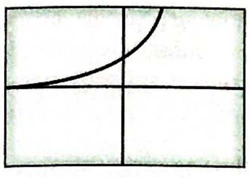
14. $y = 3^{-x}$

15. $y = -3^{-x}$

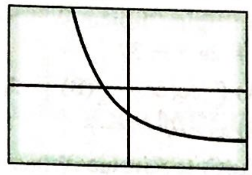
16. $y = -0.5^{-x}$

17. $y = 2^{-x} - 2$

18. $y = 1.5^x - 2$



(a)



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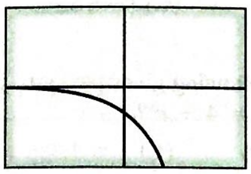

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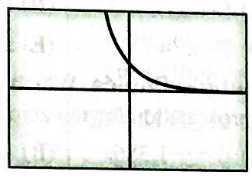
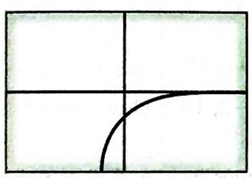
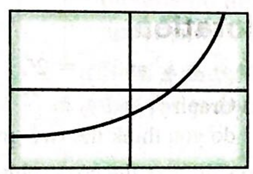


(d)



(e)



(f)

In Exercises 19–32, use an exponential model to solve the problem.

19. *Population Growth* The population of Knoxville is 500,000 and is increasing at the rate of 3.75% each year. Approximately when will the population reach 1 million?

20. *Population Growth* The population of Silver Run in the year 1890 was 6250. Assume the population increased at a rate of 2.75% per year.

(a) Estimate the population in 1915 and 1940.

(b) Approximately when did the population reach 50,000?

21. *Half-life* The approximate half-life of titanium-44 is 63 years. How long will it take a sample to lose

(a) 50% of its titanium-44?

(b) 75% of its titanium-44?

22. *Half-life* The amount of silicon-32 in a sample will decay from 28 grams to 7 grams in approximately 340 years. What is the approximate half-life of silicon-32?

23. *Radioactive Decay* The half-life of phosphorus-32 is about 14 days. There are 6.6 grams present initially.

(a) Express the amount of phosphorus-32 remaining as a function of time t.

(b) When will there be 1 gram remaining?

24. *Finding Time* If John invests $2300 in a savings account with a 6% interest rate compounded annually, how long will it take until John's account has a balance of $4150?

25. *Doubling Your Money* Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded annually.

26. *Doubling Your Money* Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded monthly.

27. *Doubling Your Money* Determine how much time is required for an investment to double in value if interest is earned at the rate of 6.25% compounded continuously.

28. *Tripling Your Money* Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded annually.

29. *Tripling Your Money* Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded daily.

30. *Tripling Your Money* Determine how much time is required for an investment to triple in value if interest is earned at the rate of 5.75% compounded continuously.

31. *Cholera Bacteria* Suppose that a colony of bacteria starts with 1 bacterium and doubles in number every half hour. How many bacteria will the colony contain at the end of 24 h?

32. *Eliminating a Disease* Suppose that in any given year, the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take

(a) to reduce the number of cases to 1000?

(b) to eliminate the disease; that is, to reduce the number of cases to less than 1?

Group Activity In Exercises 33–36, copy and complete the table for the function.

33. $y = 2x - 3$

x	y	Change (Δy)
1	?	?
2	?	?
3	?	?
4	?	

34. $y = -3x + 4$

x	y	Change (Δy)
1	?	?
2	?	?
3	?	?
4	?	

35. $y = x^2$

x	y	Change (Δy)
1	?	?
2	?	?
3	?	?
4	?	

36. $y = 3e^x$

x	y	Ratio (y_i/y_{i-1})
1	?	
		?
2	?	
		?
3	?	
		?
4	?	

37. Writing to Learn Explain how the change Δy is related to the slopes of the lines in Exercises 33 and 34. If the changes in x are constant for a linear function, what would you conclude about the corresponding changes in y?

38. *Bacteria Growth* The number of bacteria in a petri dish culture after t hours is

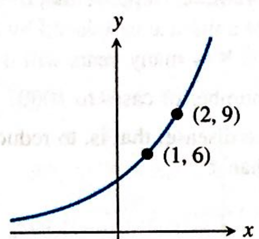
$$B = 100e^{0.693t}.$$

(a) What was the initial number of bacteria present?

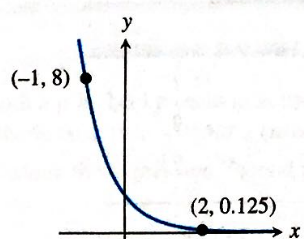
(b) How many bacteria are present after 6 hours?

(c) Approximately when will the number of bacteria be 200? Estimate the doubling time of the bacteria.

39. The graph of the exponential function $y = a \cdot b^x$ is shown below. Find a and b.



40. The graph of the exponential function $y = a \cdot b^x$ is shown below. Find a and b.



Standardized Test Questions

You may use a graphing calculator to solve the following problems.

41. True or False The number 3^{-2} is negative. Justify your answer.

42. True or False If $4^3 = 2^a$, then $a = 6$. Justify your answer.

43. Multiple Choice John invests $200 at 4.5% compounded annually. About how long will it take for John's investment to double in value?

(A) 6 yr (B) 9 yr (C) 12 yr (D) 16 yr (E) 20 yr

44. Multiple Choice Which of the following gives the domain of $y = 2e^{-x} - 3$?

(A) $(-\infty, \infty)$ (B) $[-3, \infty)$ (C) $[-1, \infty)$ (D) $(-\infty, 3]$ (E) $x \neq 0$

45. Multiple Choice Which of the following gives the range of $y = 4 - 2^{2x}$?

(A) $(-\infty, \infty)$ (B) $(-\infty, 4)$ (C) $[-4, \infty)$ (D) $(-\infty, 4]$ (E) all reals

46. Multiple Choice Which of the following gives the best approximation for the zero of $f(x) = 4 - e^x$?

(A) $x = -1.386$ (B) $x = 0.386$ (C) $x = 1.386$ (D) $x = 3$ (E) There are no zeros.

Exploration

47. Let $y_1 = x^2$ and $y_2 = 2^x$.

(a) Graph y_1 and y_2 in $[-5, 5]$ by $[-2, 10]$. How many times do you think the two graphs cross?

(b) Compare the corresponding changes in y_1 and y_2 as x changes from 1 to 2, 2 to 3, and so on. How large must x be for the changes in y_2 to overtake the changes in y_1?

(c) Solve for x: $x^2 = 2^x$.

(d) Solve for x: $x^2 < 2^x$.

Extending the Ideas

In Exercises 48 and 49, assume that the graph of the exponential function $f(x) = k \cdot a^x$ passes through the two points. Find the values of a and k.

48. $(1, 4.5), (-1, 0.5)$

49. $(1, 1.5), (-1, 6)$

Quick Quiz for AP* Preparation: Sections 1.1–1.3

You may use a graphing calculator to solve the following problems.

1. Multiple Choice Which of the following gives an equation for the line through $(3, -1)$ and parallel to the line $y = -2x + 1$?

(A) $= \frac{1}{2}x + \frac{7}{2}$ (B) $y = \frac{1}{2}x - \frac{5}{2}$ (C) $y = -2x + 5$

(D) $y = -2x - 7$ (E) $y = -2x + 1$

2. Multiple Choice If $f(x) = x^2 + 1$ and $g(x) = 2x - 1$, which of the following gives $(f \circ g)(2)$?

(A) 2 (B) 5 (C) 9 (D) 10 (E) 15

3. Multiple Choice The half-life of a certain radioactive substance is 8 hr. There are 5 grams present initially. Which of the following gives the best approximation when there will be 1 gram remaining?

(A) 2 (B) 10 (C) 15 (D) 16 (E) 19

4. Free Response Let $f(x) = e^{-x} - 2$.

(a) Find the domain of f.

(b) Find the range of f.

(c) Find the zeros of f.